Differentiation by Rule.

Rule:

If $y = x^n$ **then** $\frac{dy}{dx} = nx^{n-1}$

And $y = ax^n$ **then** $\frac{dy}{dx} = nax^{n-1}$

Note: **The derivative of a constant = 0**
And $x^0 = 1$

Example:

Differentiate each of the following with respect to x:

(i) $y = x^4$ ***(ii)*** $y = -3x^5$ ***(iii)*** $y = \frac{1}{x^2}$ ***(iv)*** $y = 3x^2 + x - 4$ ***(v)*** $y = 6$

Answer:

(i) $y = x^4$ $\frac{dy}{dx} = 3 \times x^{4-1} = 3x^3$

(ii) $y = -3x^5$ $\frac{dy}{dx} = 5 \times -3x^{5-1} = -15x^4$

(iii) $y = \frac{1}{x^2}$ x^{-2} **is the same as** $\frac{1}{x^2}$ **so** $\frac{dy}{dx} = -2 \times x^{-2-1} = -2x^{-3}$ **or** $-\frac{2}{x^3}$

(iv) $y = 3x^2 + x - 4$

$$\frac{dy}{dx} = 2 \times 3x^{2-1} + 1 \times x^{1-1} - 0 \times 4^{0-1}$$

$$\frac{dy}{dx} = 6x^1 + 1$$

(v) $y = 6$ $\frac{dy}{dx} = 0 \times 6^{0-1} = 0$

Evaluating Derivatives.

Example:

If $y = 4x^2 - 5x + 3$, *find* $\frac{dy}{dx}$ *when* $x = 2$

Answer:

$y = 4x^2 - 5x + 3,$

$$\frac{dy}{dx} = 8x - 5$$

$$\left.\frac{dy}{dx}\right|_{x = 2} = 8(2) - 5 \quad = \quad 16 - 5 \quad = \quad \underline{\underline{11}}$$

Product Rule:

Suppose *u* and *v* are functions of *x*

If $y = u.v$

Then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Example:

(i) If $y = (3x^2 + x - 4)(x^2 + 2)$, find $\frac{dy}{dx}$

(ii) Evaluate $\frac{dy}{dx}$ when $x = -2$

Answer:

(i) $y = \underbrace{(3x^2 + x - 4)}_{u}\underbrace{(x^2 + 2)}_{v}$

$u = 3x^2 + x - 4$

$\frac{du}{dx} = 6x + 1$

$v = x^2 + 2$

$\frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (3x^2 + x - 4)(2x) + (x^2 + 2)(6x + 1)$$

$$\frac{dy}{dx} = 6x^3 + 2x^2 - 8x + 6x^3 + x^2 + 12x + 2$$

$$\frac{dy}{dx} = \underline{12x^3 + 3x^2 + 4x + 2}$$

(ii) $\left.\frac{dy}{dx}\right|_{x = -2} = 12(-2)^3 + 3(-2)^2 + 4(-2) + 2$

$= 12(-8) + 3(4) + 4(-2) + 2$

$= -96 + 12 - 8 + 2$

$= \underline{-90}$

Quotient Rule:

Suppose *u* and *v* are functions of *x*

If $$y = \frac{u}{v}$$

Then $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example:

$$Find\ \frac{dy}{dx}\ \ if\ \ y = \frac{3x^2 + 2}{x + 1}$$

Answer:

$$y = \frac{3x^2 + 2}{x + 1}$$ → **u = $3x^2 + 2$** **and** **v = $x + 1$**

so $$\frac{du}{dx} = 6x \qquad \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x+1)(6x) - (3x^2 + 2)(1)}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{6x^2 + 6x - 3x^2 - 2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{3x^2 + 6x - 2}{(x+1)^2}$$

Chain Rule:

The chain rule is used when a function is raised to a power.

If y = (function)^{n}

Then $$\frac{dy}{dx} = n\ (\text{function})^{n-1}\ (\text{derivative of function})$$

Example:

Find $\frac{dy}{dx}$ ***if*** $y = (x^2 - 4x)^3$

Answer:

$$\frac{dy}{dx} = n\ (\text{function})^{n-1}\ (\text{derivative of function})$$

$$\frac{dy}{dx} = 3(x^2 - 4x)^2\,(2x - 4)$$

Finding the Slope and Equation of a Tangent to a Curve at a Point (x_1, y_1).

Steps:

1. **Find** $\frac{dy}{dx}$
2. **Evaluate** $\left.\frac{dy}{dx}\right|_{x = x1}$ **(This gives the slope)**
3. **Use the equation of the line formula with the point and slope.**

Example:

Find the equation of the tangent to the curve $y = x^2 - 3x + 2$ ***at the point (1, 0).***

Answer:

1. $y = x^2 - 3x + 2$

$\frac{dy}{dx} = 2x - 3$

2. **The point is (1, 0) so** $x = 1$

$$\left.\frac{dy}{dx}\right|_{x = 1} = 2(1) - 3$$

$$= 2 - 3$$

$$= - 1$$

Slope = - 1

3. $m = - 1 \qquad x_1 = 1 \qquad y_1 = 0$

$$y - y_1 = m\,(x - x_1)$$

$$y - 0 = - 1\,(x - 1)$$

$$y = -x + 2$$

$\underline{x + y = 2}$

Finding the coordinates of a point on the curve when given $\frac{dy}{dx}$.

Example:

Find the coordinates of a points to the curve $y = 2x^3 - 3x^2 - 12x$ at which the tangents to the curve are parallel to the line $y = 24x + 3$

Answer:

Firstly, the slope of the line $y = 24x + 3$ is 24. (remember $y = mx + c$)

Step 1: $y = 2x^3 - 3x^2 - 12x$

$$\frac{dy}{dx} = 6x^2 - 6x - 12$$

Step 2: **Let** $\frac{dy}{dx}$ **= slope**

$6x^2 - 6x - 12 = 24$

$6x^2 - 6x - 36 = 0$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

So $x = 3$ **and** $x = -2$

Step 3: **Find y values.**

When $x = 3$	**When $x = -2$**
$y = 2(3)^3 - 3(3)^2 - 12(3)$	$y = 2(-2)^3 - 3(-2)^2 - 12(-2)$
$y = 54 - 27 - 36$	$y = -16 - 12 + 24$
$y = -9$	$y = -4$
Point = (3, -9)	**Point = (-2, -4)**

Maximum and Minimum Points.

A maximum or minimum point occurs when $\frac{dy}{dx} = 0$

Example:

Find, using calculus, the local maximum and local minimum points of the curve $y = 2x^3 - 3x^2 - 12x + 18$

Answer:

$y = 2x^3 - 3x^2 - 12x + 18$

1. $\frac{dy}{dx} = 6x^2 - 6x - 12$

2. **Let** $\frac{dy}{dx} = 0$

$6x^2 - 6x - 12 = 0$ (

(divide by 6 to simplify) $x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

so $x = 2$ **and** $x = -1$

3. Find the y values

When $x = 2$	**When** $x = -1$
$y = 2x^3 - 3x^2 - 12x + 18$	$y = 2x^3 - 3x^2 - 12x + 18$
$y = 2(2)^3 - 3(2)^2 - 12(2) + 18$	$y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 18$
$y = 16 - 12 - 24 + 18$	$y = -2 - 3 + 12 + 18$
$y = -2$	$y = 25$
Point = (2, -2)	**Point = (-1, 25)**

4. 25 is greater than -2, so the Maximum Point = (-1, 25) and the Minimum Point = (2, -2)

Increasing or Decreasing

When the curve is increasing, the slope of the tangent to the curve will be positive. When it is decreasing, the slope will be negative.

Example:

Rate of Change.

1. **Velocity (speed) = v =** $\dfrac{ds}{dt}$

2. **Acceleration = a =** $\dfrac{d^2s}{dt^2}$

Example:

A particle moves along a straight line such that, after t seconds, the distance s metres from a fixed point o is given by s(t) = $t^3 - 9t^2 + 24t$

Find (i) the speed and velocity of the particle in terms of t.

(ii) the speed of the particle after 6 seconds

(iii) the times when the speed is zero

(iv) the acceleration of the particle after 4 seconds

(v) the time at which the acceleration is zero

(vi) the time at which the acceleration is 6 m/s^2

Answer: $s(t) = t^3 - 9t^2 + 24t$

(i) $\dfrac{ds}{dt} = 3t^2 - 18t + 24$ **and** $\dfrac{d^2s}{dt^2} = 6t - 18$

(ii) $\left.\dfrac{ds}{dt}\right|_{t=6} = 3(6)^2 - 18(6) + 24 = 24\text{m/s}$

(iii) $3t^2 - 18t + 24 = 0$

$t^2 - 6t + 8 = 0$

$(t - 4)(t - 2) = 0$

$t = 4\text{sec}$ **and** $t = 2\text{sec}$

(iv) $\dfrac{d^2s}{dt^2} = 6(4) - 18 = 24 - 18 = 6 \text{ m/s}^2$

(v) $6t - 18 = 0$

$6t = 18$

$t = 3$ **sec.**

(vi) $6t - 18 = 6$

$6t = 24$

$t = 4\text{sec}$